2.3 連続確率分布.

* ベータ分布.

$\mu \in (0.1)$: 変数を生成.

pdf. $\mathrm{Beta}(\mu | a.b) = C_B(a.b)\mu^{a-1}(1-\mu)^{b-1} \quad (a.b \in \mathbb{R}_{>0})$

$$\int_0^1 \mathrm{Beta}(\mu|a.b)\,d\mu = 1$$ より

$$\underbrace{C_B(a.b)\int_0^1 \mu^{a-1}(1-\mu)^{b-1}}_{} = 1.$$

↓ベータ関数の定義.

$$C_B(a.b)B(a.b) = 1. \quad \therefore C_B(a.b) = \frac{1}{B(a.b)}$$

$$B(a.b) = \frac{\Gamma(a)\Gamma(b)}{\Gamma(a+b)}$$ という性質がある.

・ 対数表示:

$$\log \mathrm{Beta}(\mu|a.b) = (a-1)\log\mu + (b-1)\log(1-\mu) + \log C_B(a.b)$$

・ 平均.

$$\mathbb{E}[X] = \int_0^1 x\, C_B(a.b)\, x^{a-1}(1-x)^{b-1}dx$$

$$= C_B(a.b)\int_0^1 x^a(1-x)^{b-1}dx = C_B(a,b)B(a+1,b)$$

$$= \frac{\Gamma(a+b)}{\Gamma(a)\cancel{\Gamma(b)}}\frac{\Gamma(a+1)\cancel{\Gamma(b)}}{\Gamma(a+b+1)} = \frac{\cancel{\Gamma(a+b)}\cdot a\cancel{\Gamma(a)}}{\cancel{\Gamma(a)}\cdot(a+b)\cancel{\Gamma(a+b)}} = \frac{a}{a+b}.$$

ガンマ関数の性質

$\Gamma(x+1) = x\Gamma(x)$.

$$\mathbb{E}[\log X] = C_B(a,b)\int_0^1 (\log x)\, x^{a-1}(1-x)^{b-1}\,dx.$$

ベータ関数の形にもっていきたいけど $\log x$ がじゃま.

ところで,

$$\frac{d}{da}x^a = \frac{d}{da}\left(e^{\log x}\right)^a = \frac{d}{da}e^{a\log x}$$

$$= e^{a\log x}\cdot \log x$$

$$= x^a \log x.$$

微分の定義

$$f'(a) = \lim_{h\to 0}\frac{f(a+h)-f(a)}{h}$$

を用いてもOK. めんどいけど

これを用いると,

$$\mathbb{E}[\log X] = C_B(a,b)\int_0^1 \frac{\partial}{\partial a}x^{a-1}(1-x)^{b-1}\,dx$$

微分と積分の交換はここではやってもよい.

$$= C_B(a,b)\frac{\partial}{\partial a}\int_0^1 x^{a-1}(1-x)^{b-1}\,dx$$

$$= C_B(a,b)\frac{\partial}{\partial a}\left(\frac{1}{C_B(a,b)}\int_0^1 \mathrm{Beta}(x\,|\,a,b)\,dx\right)$$

$$= C_B(a,b)\frac{\partial}{\partial a}\left(\frac{1}{C_B(a,b)}\right) = \frac{1}{B(a,b)}\frac{\partial}{\partial a}B(a,b)$$

$$\frac{d}{dx}\log f(x) = \frac{\frac{d}{dx}f(x)}{f(x)}.$$

$$= \frac{\partial}{\partial a}\left(\log B(a,b)\right)$$

$$= \frac{\partial}{\partial a}\left(\log\Gamma(a) + \log\Gamma(b) - \log\Gamma(a+b)\right)$$

$$= \psi(a) - \psi(a+b). \qquad \left(\psi(x) = \frac{d}{dx}\log\Gamma(x)\right)$$

同様に,

$$\mathbb{E}[\log(1-X)] = \psi(b) - \psi(a+b).$$

・エントロピー

$$H[\mathrm{Beta}(\mu|a,b)] = -\mathbb{E}[\log \mathrm{Beta}(\mu|a,b)]$$

$$= -(a-1)\mathbb{E}[\log\mu] - (b-1)\mathbb{E}[\log(1-\mu)] - \log C_B(a,b)$$

$$= -(a-1)\psi(a) - (b-1)\psi(b) + (a+b-2)\psi(a+b) - \log C_B(a,b).$$

* Dirichlet分布.

$\pi = \begin{pmatrix} \pi_1 \\ \vdots \\ \pi_K \end{pmatrix}$ K-dim. Prob. vec. を生成.

pdf. $\mathrm{Dir}(\pi|\alpha) = C_D(\alpha)\prod_{k=1}^{K} \pi_k^{\alpha_k - 1}$. $\alpha > 0$.

$$C_D(\alpha) = \frac{\Gamma(\sum_{k=1}^{K}\alpha_k)}{\prod_{k=1}^{K}\Gamma(\alpha_k)}$$

◦ Dirichlet分布が確率分布であることの確認.

π : prob. vec. なので, $\pi_i \geq 0$ $(i = 1, \cdots, K-1)$ が決まると,

$\pi_K = 1 - \sum_{i=1}^{K-1}\pi_i$ から決定される. よって,

$D_K := \{\pi \in \mathbb{R}^K \mid \pi_i \geq 0 \ (i = 1, \cdots, K), \ \sum_{i=1}^{K}\pi_i \leq 1\}$ として

D_{K-1} で $\mathrm{Dir}(\pi|\alpha)$ を積分する. $d\pi_1 \cdots d\pi_{K-1} =: d\pi_{1:K-1}$ などとかく

$$\int_{D_{K-1}} \mathrm{Dir}(\pi|\alpha)\, d\pi_{1:K-1} = \frac{\Gamma(\sum_{k=1}^{K}\alpha_k)}{\prod_{k=1}^{K}\Gamma(\alpha_k)} \int_{D_{K-1}} \prod_{k=1}^{K} \pi_k^{\alpha_k - 1}\, d\pi_{1:K-1}.$$

あとは, 以下の主張を示せばよい.

[Claim] $\displaystyle \int_{D_{K-1}} \prod_{k=1}^{K} \pi_k^{\alpha_k - 1} d\pi_{1:K-1} = \frac{\prod_{k=1}^{K} \Gamma(\alpha_k)}{\Gamma(\sum_{k=1}^{K} \alpha_k)}$.

pf. $\displaystyle \prod_{k=1}^{K} \Gamma(\alpha_k) = \prod_{k=1}^{K} \int_0^\infty t_k^{\alpha_k - 1} e^{-t_k} dt_k$

$$= \int_0^\infty \cdots \int_0^\infty \prod_{k=1}^{K} \left(t_k^{\alpha_k - 1} e^{-t_k} \right) dt_k .$$

ここで変換変換 $(t_1, \cdots, t_K) \to (\pi_1, \cdots, \pi_{K-1}, x)$ を次のようにする：

$$t_i = \pi_i x \quad (i = 1, \cdots, K-1), \quad t_K = \left(1 - \sum_{i=1}^{K-1} \pi_i\right) x . \quad -(*)$$

$\displaystyle 1 - \sum_{i=1}^{K-1} \pi_i =: \pi_K$ とおく.

積分範囲 $[0, \infty)^K$ がどのように変換されるかをみる：

$t_i \geq 0 \ (i = 1, \cdots, K)$ より

$\pi_i \geq 0$ and $\displaystyle 1 - \sum_{i=1}^{K-1} \pi_i \geq 0$ and $x \geq 0$ として考えればよい.

($\pi_i \leq 0$ and $\displaystyle 1 - \sum_{i=1}^{K-1} \pi_i \leq 0$ and $x \leq 0$ も考えられるが,

改めて $\pi_i' = -\pi_i$, $\displaystyle 1 - \sum_{i=1}^{K-1} \pi_i' = -\left(1 - \sum_{i=1}^{K-1} \pi_i\right)$, $x' = -x$

と変数変換すればこの Jacobian の絶対値は 1 なので OK.)

ゆえに, 積分範囲は, $S := \underbrace{D_{K-1}}_{\pi_i\text{たち}} \times \underbrace{[0, \infty)}_{x}$ となる.

また, 変数変換 $(*)$ の Jacobian は 次のとおり.

$$J(\pi_1, \cdots, \pi_{K-1}, x) = \det \frac{\partial(t_1, \cdots, t_K)}{\partial(\pi_1, \cdots, \pi_{K-1}, x)}$$

$$= \det \left(\begin{array}{cccc|c} x & & & 0 & \pi_1 \\ & x & & & \pi_2 \\ & & \ddots & & \vdots \\ 0 & & & x & \pi_{K-1} \\ \hline -x & -x & \cdots & -x & \pi_K \end{array} \right) = x^{K-1} \ (\geq 0).$$

$$\therefore \prod_{k=1}^{K} \Gamma(\alpha_k)$$

$$= \int_S \prod_{k=1}^{K} \left((\pi_k x)^{\alpha_k - 1} e^{-\pi_k x} \right) \underbrace{x^{K-1}}_{\text{Jacobianの絶対値}} d\pi_{1:K-1}\, dx$$

$$= \int_{D_{K-1}} \prod_{k=1}^{K} \pi_k^{\alpha_k - 1} d\pi_{1:K-1} \times \int_0^{\infty} x^{\sum_{k=1}^{K} \alpha_k - K + K - 1} e^{-\underbrace{\sum_{k=1}^{K} \pi_k}_{=1} x} dx$$

$$= \int_{D_{K-1}} \prod_{k=1}^{K} \pi_k^{\alpha_k - 1} d\pi_{1:K-1} \times \Gamma\left(\sum_{k=1}^{K} \alpha_k \right). \qquad \blacksquare$$

- 対数表示:

$$\log \mathrm{Dir}(\pi | \alpha) = \sum_{k=1}^{K} (\alpha_k - 1) \log \pi_k + \log C_D(\alpha).$$

- 平均.

$$\mathbb{E}[\pi] = \int \pi \, C_D(\alpha) \prod_{k=1}^{K} \pi_k^{\alpha_k - 1} d\pi_{1:K-1}$$

第k要素に注目.

$$\mathbb{E}[\pi_k] = \int \pi_k \, C_D(\alpha) \prod_{k'=1}^{K} \pi_{k'}^{\alpha_{k'} - 1} d\pi_{1:K-1}$$

$$= C_D(\alpha)\int \pi_1^{\alpha_1-1}\cdots\pi_{k-1}^{\alpha_{k-1}-1}\pi_k^{\alpha_k}\pi_{k+1}^{\alpha_{k+1}-1}\cdots\pi_K^{\alpha_K-1}\,d\pi_{1:K-1}$$

$$= \frac{C_D(\alpha)}{C_D(\alpha_k)}\int \mathrm{Dir}(\pi\,|\,\alpha_k)\,d\pi_{1:K-1} \quad (\alpha_k = \alpha + e_k)$$

$$= \frac{\dfrac{\Gamma\left(\sum_{k=1}^{K}\alpha_k\right)}{\prod_{k=1}^{K}\Gamma(\alpha_k)}}{\dfrac{\Gamma\left(\sum_{k=1}^{K}\alpha_k+1\right)}{\prod_{k'\neq k}\Gamma(\alpha_{k'})\Gamma(\alpha_k+1)}} = \frac{\alpha_k}{\sum_{k=1}^{K}\alpha_k}.$$

$$\therefore\ \mathbb{E}[\pi] = \frac{1}{\sum_{k=1}^{K}\alpha_k}\alpha.$$

$\log\pi = \begin{pmatrix}\log\pi_1\\ \vdots\\ \log\pi_K\end{pmatrix}$ と略記.

- $\mathbb{E}[\log\pi] = \int \log\pi\, C_D(\alpha)\prod_{k=1}^{K}\pi_k^{\alpha_k-1}\,d\pi_{1:K-1}$

第k要素に注目.

$$\mathbb{E}[\log\pi_k] = \int \log\pi_k\, C_D(\alpha)\prod_{k'=1}^{K}\pi_{k'}^{\alpha_{k'}-1}\,d\pi_{1:K-1}$$

$$= C_D(\alpha)\int \log\pi_k \prod_{k'=1}^{K}\pi_{k'}^{\alpha_{k'}-1}\,d\pi_{1:K-1}$$

$$= C_D(\alpha)\int \pi_1^{\alpha_1-1}\cdots\pi_{k-1}^{\alpha_{k-1}-1}\left(\frac{\partial}{\partial\alpha_k}\pi_k^{\alpha_k-1}\right)\pi_{k+1}^{\alpha_{k+1}-1}\cdots\pi_K^{\alpha_K-1}\,d\pi_{1:K-1}$$

$$= C_D(\alpha)\frac{\partial}{\partial\alpha_k}\left(\int\prod_{k'=1}^{K}\pi_{k'}^{\alpha_{k'}-1}\,d\pi_{1:K-1}\right)$$

$$= C_D(\alpha)\frac{\partial}{\partial\alpha_k}\left(\frac{1}{C_D(\alpha)}\int C_D(\alpha)\prod_{k'=1}^{K}\pi_{k'}^{\alpha_{k'}-1}\,d\pi_{1:K-1}\right)$$

$$= C_D(\alpha)\frac{\partial}{\partial\alpha_k}\left(\frac{1}{C_D(\alpha)}\int \mathrm{Dir}(\pi|\alpha)d\pi_{1:K-1}\right)$$
$$= C_D(\alpha)\frac{\partial}{\partial\alpha_k}\left(\frac{1}{C_D(\alpha)}\right)$$
$$= \frac{\partial}{\partial\alpha_k}\log\frac{1}{C_D(\alpha)}$$
$$= \frac{\partial}{\partial\alpha_k}\log\frac{\prod_{k'=1}^{K}\Gamma(\alpha_{k'})}{\Gamma\left(\sum_{k'=1}^{K}\alpha_{k'}\right)}$$
$$= \frac{\partial}{\partial\alpha_k}\left(\sum_{k'=1}^{K}\log\Gamma(\alpha_{k'}) - \log\Gamma\left(\sum_{k'=1}^{K}\alpha_{k'}\right)\right)$$
$$= \frac{\partial}{\partial\alpha_k}\log\Gamma(\alpha_k) - \frac{\partial}{\partial\alpha_k}\log\Gamma\left(\sum_{k'=1}^{K}\alpha_{k'}\right)$$
$$= \psi(\alpha_k) - \psi\left(\sum_{k'=1}^{K}\alpha_{k'}\right).$$
$$\therefore\ \mathbb{E}[\log\pi] = \psi(\alpha) - \psi\left(\sum_{k=1}^{K}\alpha_k\right)$$

- エントロピー.

$$H[\mathrm{Dir}(\pi|\alpha)] = \mathbb{E}[-\log\mathrm{Dir}(\pi|\alpha)]$$
$$= -\sum_{k=1}^{K}(\alpha_k-1)\mathbb{E}[\log\pi_k] - \log C_D(\alpha)$$
$$= -\sum_{k=1}^{K}(\alpha_k-1)\left(\psi(\alpha_k) - \psi\left(\sum_{k'=1}^{K}\alpha_{k'}\right)\right) - \log C_D(\alpha).$$

- KL divergence.

$p(\pi) = \mathrm{Dir}(\pi|\alpha)$, $q(\pi) = \mathrm{Dir}(\pi|\hat{\alpha})$ とする.

$$\mathrm{KL}[q\|p] = -\mathrm{H}[\mathrm{Dir}(\pi|\hat{\alpha})] - \mathbb{E}_q[\log \mathrm{Dir}(\pi|\alpha)].$$

$$\mathbb{E}_q[\log \mathrm{Dir}(\pi|\alpha)] = \sum_{k=1}^{K}(\alpha_k - 1)\mathbb{E}_q[\log \pi_k] + \log C_D(\alpha)$$

$$= \sum_{k=1}^{K}(\alpha_k - 1)\left(\psi(\hat{\alpha}_k) - \psi\left(\sum_{k'=1}^{K}\hat{\alpha}_{k'}\right)\right) + \log C_D(\alpha)$$

$\therefore$ $\mathrm{KL}[q\|p]$

$$= \sum_{k=1}^{K}(\hat{\alpha}_k - 1)\left(\psi(\hat{\alpha}_k) - \psi\left(\sum_{k'=1}^{K}\hat{\alpha}_{k'}\right)\right) + \log C_D(\hat{\alpha})$$

$$- \sum_{k=1}^{K}(\alpha_k - 1)\left(\psi(\hat{\alpha}_k) - \psi\left(\sum_{k'=1}^{K}\hat{\alpha}_{k'}\right)\right) - \log C_D(\alpha)$$

$$= \sum_{k=1}^{K}(\hat{\alpha}_k - \alpha_k)\left(\psi(\hat{\alpha}_k) - \psi\left(\sum_{k'=1}^{K}\hat{\alpha}_{k'}\right)\right) + \log\frac{C_D(\hat{\alpha})}{C_D(\alpha)}.$$

* ガンマ分布.

$\lambda \in \mathbb{R}_{>0}$ を生成する分布.

pdf. $\mathrm{Gam}(\lambda|a, b) = C_G(a, b)\lambda^{a-1}e^{-b\lambda}$,

$$C_G(a, b) = \frac{b^a}{\Gamma(a)}. \quad (a, b \in \mathbb{R}_{>0}).$$

・対数表示.

$$\log \mathrm{Gam}(\lambda|a, b) = (a-1)\log\lambda - b\lambda + \log C_G(a, b).$$

・平均.

$$\mathbb{E}[\lambda] = \int_0^{\infty}\lambda \cdot C_G(a, b)\lambda^{a-1}e^{-b\lambda}\,d\lambda$$

$$= C_G(a,b)\int_0^\infty \lambda^a e^{-b\lambda}\,d\lambda$$

$$= C_G(a,b)\,\frac{1}{C_G(a+1,b)}\int_0^\infty C_G(a+1,b)\,\lambda^a e^{-b\lambda}\,d\lambda$$

$$= C_G(a,b)\,\frac{1}{C_G(a+1,b)}\int_0^\infty \mathrm{Gam}(\lambda\mid a+1,b)\,d\lambda$$

$$= \frac{\dfrac{b^a}{\Gamma(a)}}{\dfrac{b^{a+1}}{\Gamma(a+1)}} = \frac{a\cancel{\Gamma(a)}}{\cancel{\Gamma(a)}}\,\frac{1}{b} = \frac{a}{b}.$$

$$\mathbb{E}[\log\lambda] = \int_0^\infty \log\lambda\cdot C_G(a,b)\,\lambda^{a-1}e^{-b\lambda}\,d\lambda$$

$$= C_G(a,b)\int_0^\infty \left(\frac{\partial}{\partial a}\lambda^{a-1}\right)e^{-b\lambda}\,d\lambda$$

$$= C_G(a,b)\,\frac{\partial}{\partial a}\int_0^\infty \lambda^{a-1}e^{-b\lambda}\,d\lambda$$

$$= C_G(a,b)\,\frac{\partial}{\partial a}\,\frac{1}{C_G(a,b)}\underbrace{\int_0^\infty \mathrm{Gam}(\lambda\mid a,b)\,d\lambda}_{=1}$$

$$= C_G(a,b)\,\frac{\partial}{\partial a}\,\frac{1}{C_G(a,b)}$$

$$= \frac{\partial}{\partial a}\log\frac{1}{C_G(a,b)}$$

$$= \frac{\partial}{\partial a}\left(\log\Gamma(a) - a\log b\right)$$

$$= \psi(a) - \log b.$$

・ エントロピー.

$$H[\mathrm{Gam}(\lambda|a,b)]$$
$$= -\mathbb{E}[\log \mathrm{Gam}(\lambda|a,b)]$$
$$= -\mathbb{E}[\log C_G(a,b) + (a-1)\log\lambda - b\lambda]$$
$$= -(a-1)\mathbb{E}[\log\lambda] + b\mathbb{E}[\lambda] - \log C_G(a,b)$$
$$= (1-a)\psi(a) - (1-a)\log b + b\cdot\frac{a}{b} - a\log b + \log\Gamma(a)$$
$$= (1-a)\psi(a) - \log b + a + \log\Gamma(a).$$

・ $p(\lambda) = \mathrm{Gam}(\lambda|a,b)$ と $q(\lambda) = \mathrm{Gam}(\lambda|\hat{a},\hat{b})$ の KL-divergence.

$$\mathrm{KL}[q(\lambda)\|p(\lambda)] = -H[\mathrm{Gam}(\lambda|\hat{a},\hat{b})] - \mathbb{E}_q[\log \mathrm{Gam}(\lambda|a,b)].$$

$$\mathbb{E}_q[\log \mathrm{Gam}(\lambda|a,b)]$$
$$= (a-1)\mathbb{E}_q[\log\lambda] - b\mathbb{E}_q[\lambda] + \log C_G(a,b)$$
$$= (a-1)(\psi(\hat{a}) - \log\hat{b}) - \frac{b\hat{a}}{\hat{b}} + a\log b - \log\Gamma(a).$$

$$\therefore \mathrm{KL}[q(\lambda)\|p(\lambda)]$$
$$= -(1-\hat{a})\psi(\hat{a}) + \log\hat{b} - \hat{a} - \log\Gamma(\hat{a})$$
$$-(a-1)(\psi(\hat{a}) - \log\hat{b}) + \frac{b\hat{a}}{\hat{b}} - a\log b + \log\Gamma(a).$$
$$= (\hat{a}-a)\psi(\hat{a}) - a\log\frac{b}{\hat{b}} + \hat{a}\left(\frac{b}{\hat{b}} - 1\right) + \log\frac{\Gamma(a)}{\Gamma(\hat{a})}.$$

＊ 1次元Gauss分布.

$x \in \mathbb{R}$ を生成.

pdf. $$\mathcal{N}(x|\mu,\sigma^2) = \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left(-\frac{1}{2\sigma^2}(x-\mu)^2\right).$$

$\mu \in \mathbb{R}$: 平均パラメータ, $\sigma^2 \in \mathbb{R}_{>0}$: 分散パラメータ.

・対数表示.

$$\log \mathcal{N}(x|\mu,\sigma^2)$$
$$= -\frac{1}{2}\left(\log 2\pi + \log\sigma^2\right) + \left(-\frac{1}{2\sigma^2}(x-\mu)^2\right)$$
$$= -\frac{1}{2}\left(\frac{1}{\sigma^2}(x-\mu)^2 + \log\sigma^2 + \log 2\pi\right).$$

← xについて上に凸の2次関数

・平均.

$$\mathbb{E}[X] = \int_{-\infty}^{\infty} x\,\mathcal{N}(x|\mu,\sigma^2)\,dx$$

$$= \int_{-\infty}^{\infty}(x-\mu)\mathcal{N}(x|\mu,\sigma^2)\,dx + \mu\underbrace{\int_{-\infty}^{\infty}\mathcal{N}(x|\mu,\sigma^2)\,dx}_{=1}$$

$$= \mu + \int_{-\infty}^{\infty}(x-\mu)\frac{1}{\sqrt{2\pi\sigma^2}}\exp\left(-\frac{1}{2\sigma^2}(x-\mu)^2\right)dx$$

$$= \mu + \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\frac{x-\mu}{\sigma}\exp\left(-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2\right)dx$$

$\swarrow$ $z = \frac{x-\mu}{\sigma}$

$$= \mu + \frac{\sigma}{\sqrt{2\pi}}\int_{-\infty}^{\infty} z e^{-\frac{z^2}{2}}\,dz.$$

$$= \mu - \frac{\sigma}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \left(\frac{d}{dz} e^{-\frac{z^2}{2}} \right) dz$$

$$= \mu - \frac{\sigma}{\sqrt{2\pi}} \left[e^{-\frac{z^2}{2}} \right]_{-\infty}^{\infty}$$

$$= \mu .$$

(別の計算方法)

$$\mathbb{E}[X] = \mu + \int_{-\infty}^{\infty} (x-\mu) \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left(-\frac{1}{2\sigma^2}(x-\mu)^2 \right) dx$$

$$= \mu + \sigma^2 \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi\sigma^2}} \frac{\partial}{\partial \mu} \exp\left(-\frac{1}{2\sigma^2}(x-\mu)^2 \right) dx$$

$$= \mu + \sigma^2 \frac{\partial}{\partial \mu} \int_{-\infty}^{\infty} \underbrace{\frac{1}{\sqrt{2\pi\sigma^2}} \exp\left(-\frac{1}{2\sigma^2}(x-\mu)^2 \right)}_{= \mathcal{N}(x|\mu,\sigma^2)} dx$$

$$= \mu + \sigma^2 \frac{\partial}{\partial \mu} 1$$

$$= \mu .$$

$$
\mathbb{E}[X^2] = \int_{-\infty}^{\infty} x^2 \mathcal{N}(x|\mu,\sigma^2)\,dx
$$

$$
= \int_{-\infty}^{\infty} (x-\mu)^2 \mathcal{N}(x|\mu,\sigma^2)\,dx + \underbrace{2\mu\int_{-\infty}^{\infty} x\,\mathcal{N}(x|\mu,\sigma^2)\,dx}_{=\mathbb{E}[X]=\mu}
$$

$$
-\mu^2 \underbrace{\int_{-\infty}^{\infty} \mathcal{N}(x|\mu,\sigma^2)\,dx}_{=1}
$$

$$
= \mu^2 + \int_{-\infty}^{\infty} (x-\mu)^2 \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left(-\frac{1}{2\sigma^2}(x-\mu)^2\right) dx
$$

$$
= \mu^2 + \frac{\sigma}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \left(\frac{x-\mu}{\sigma}\right)^2 \exp\left(-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2\right) dx
$$

$z = \frac{x-\mu}{\sigma}$

$$
= \mu^2 + \frac{\sigma^2}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \underbrace{z^2 e^{-\frac{z^2}{2}}}_{\text{偶関数}}\,dz
$$

$$
= \mu^2 + \frac{\sigma^2}{\sqrt{2\pi}}\, 2\int_{0}^{\infty} z^2 e^{-\frac{z^2}{2}}\,dz
$$

$$
= \mu^2 - \sigma^2 \sqrt{\frac{2}{\pi}} \int_0^{\infty} z \left(\frac{d}{dz} e^{-\frac{z^2}{2}}\right) dz
$$

$$
= \mu^2 - \sigma^2 \sqrt{\frac{2}{\pi}} \left(\left[z e^{-\frac{z^2}{2}} \right]_0^{\infty} - \int_0^{\infty} e^{-\frac{z^2}{2}}\,dz \right)
$$

$$
= \mu^2 + \sigma^2 \sqrt{\frac{2}{\pi}} \underbrace{\int_0^{\infty} e^{-\frac{z^2}{2}}\,dz}_{\text{Gauss積分}}
$$

$$
= \mu^2 + \sigma^2 \sqrt{\frac{2}{\pi}} \cdot \frac{1}{2}\sqrt{2\pi} = \mu^2 + \sigma^2.
$$

$$
\therefore \mathbb{V}[X] = \mathbb{E}[X^2] - \mathbb{E}[X]^2 = \sigma^2.
$$

(別の計算方法) ($\sigma^{-2}=\lambda^{-1}$とも書く.)

$$\mathbb{E}[X^2]=\mu^2+\int_{-\infty}^{\infty}(x-\mu)^2\frac{1}{\sqrt{2\pi\sigma^2}}\exp\left(-\frac{1}{2\sigma^2}(x-\mu)^2\right)dx$$

$$=\mu^2+\int_{-\infty}^{\infty}(x-\mu)^2\frac{1}{\sqrt{2\pi\sigma^2}}\exp\left(-\frac{1}{2}\lambda(x-\mu)^2\right)dx$$

$$=\mu^2-\frac{2}{\sqrt{2\pi\sigma^2}}\int_{-\infty}^{\infty}\frac{\partial}{\partial\lambda}\exp\left(-\frac{1}{2}\lambda(x-\mu)^2\right)dx$$

$$=\mu^2-\frac{2}{\sqrt{2\pi\sigma^2}}\frac{\partial}{\partial\lambda}\int_{-\infty}^{\infty}\exp\left(-\frac{1}{2}\lambda(x-\mu)^2\right)dx$$

$$=\mu^2-\frac{2}{\sqrt{\cancel{2\pi}\sigma^2}}\frac{\partial}{\partial\lambda}\sqrt{\frac{\cancel{2\pi}}{\lambda}}\underbrace{\int_{-\infty}^{\infty}\frac{1}{\sqrt{2\pi\sigma^2}}\exp\left(-\frac{1}{2\sigma^2}(x-\mu)^2\right)dx}_{=\mathcal{N}(x|\mu,\sigma^2)}$$

$$=\mu^2-\frac{\cancel{2}}{\sqrt{\sigma^2}}\cdot\left(-\frac{1}{\cancel{2}}\lambda^{-\frac{3}{2}}\right)$$

$$=\mu^2+\frac{1}{\cancel{\sqrt{\sigma^2}}}\cdot\sigma^2\cancel{\sqrt{\sigma^2}}$$

$$=\mu^2+\sigma^2.$$

・エントロピー.

$$H[\mathcal{N}(x|\mu,\sigma^2)]$$

$$=-\mathbb{E}[\log\mathcal{N}(x|\mu,\sigma^2)]$$

$$=-\mathbb{E}\left[-\frac{1}{2}\left(\frac{1}{\sigma^2}(x-\mu)^2+\log\sigma^2+\log 2\pi\right)\right]$$

$$= \frac{1}{2}\left(\frac{1}{\sigma^2}\mathbb{E}\left[(x-\mu)^2\right] + \log\sigma^2 + \log 2\pi\right)$$

$$= \frac{1}{2}\left(\frac{1}{\sigma^2}\mathbb{V}[x] + \log\sigma^2 + \log 2\pi\right)$$

$$= \frac{1}{2}\left(1 + \log\sigma^2 + \log 2\pi\right).$$ μに依らない.

・ $p(x) = \mathcal{N}(x|\mu, \sigma^2)$, $q(x) = \mathcal{N}(x|\hat{\mu}, \hat{\sigma}^2)$ の KL-divergence.

$\mathrm{KL}[q(x)\|p(x)]$

$$= -\mathrm{H}[q(x)] - \mathbb{E}_q[\log p(x)].$$

$$= -\frac{1}{2}\left(1 + \log\hat{\sigma}^2 + \log 2\pi\right) - \mathbb{E}_q[\log p(x)].$$

$\mathbb{E}_q[\log p(x)]$

$$= \mathbb{E}_q\left[-\frac{1}{2}\left(\frac{1}{\sigma^2}(x-\mu)^2 + \log\sigma^2 + \log 2\pi\right)\right]$$

$$= -\frac{1}{2}\left(\frac{1}{\sigma^2}\mathbb{E}_q\left[x^2 - 2\mu x + \mu^2\right] + \log\sigma^2 + \log 2\pi\right)$$

$$= -\frac{1}{2}\left(\frac{1}{\sigma^2}\left(\hat{\mu}^2 + \hat{\sigma}^2 - 2\mu\hat{\mu} + \mu^2\right) + \log\sigma^2 + \log 2\pi\right).$$

$$= -\frac{1}{2}\left(\frac{1}{\sigma^2}\left((\mu-\hat{\mu})^2 + \hat{\sigma}^2\right) + \log\sigma^2 + \log 2\pi\right).$$

$\therefore$ $\mathrm{KL}[q(x)\|p(x)]$

$$= -\frac{1}{2}\left(1 + \log\hat{\sigma}^2 + \cancel{\log 2\pi}\right)$$

$$+ \frac{1}{2}\left(\frac{1}{\sigma^2}\left((\mu-\hat{\mu})^2 + \hat{\sigma}^2\right) + \log\sigma^2 + \cancel{\log 2\pi}\right).$$

$$= \frac{1}{2}\left(\frac{1}{\sigma^2}\left((\mu-\hat{\mu})^2 + \hat{\sigma}^2\right) + \log\frac{\sigma^2}{\hat{\sigma}^2} - 1\right).$$

Remark Gauss積分について.

- $a>0,\ b\in\mathbb{R}$として,

$$I(a) := \int_{-\infty}^{\infty} e^{-a(x-b)^2}dx = \sqrt{\frac{\pi}{a}}$$

pf. $\sqrt{a}(x-b)=z$と変数変換すると, $\frac{dx}{dz}=\frac{1}{\sqrt{a}}$ なので,

$$I(a) = \frac{1}{\sqrt{a}}\int_{-\infty}^{\infty} e^{-z^2}dz. \quad I := \int_{-\infty}^{\infty} e^{-z^2}dz = \sqrt{\pi}\text{を示す.}$$

$$I^2 = \left(\int_{-\infty}^{\infty} e^{-x^2}dx\right)^2 = \left(\int_{-\infty}^{\infty} e^{-x^2}dx\right)\left(\int_{-\infty}^{\infty} e^{-y^2}dy\right)$$

$$= \int_{\mathbb{R}^2} e^{-(x^2+y^2)}dxdy$$

極座標変換 $x=r\cos\theta,\ y=r\sin\theta$

Jacobianは $r>0$.

$$= \int_0^{\infty}\int_0^{2\pi} re^{-r^2}drd\theta$$

$\frac{d}{dr}e^{-r^2} = -2re^{-r^2}$

$$= 2\pi\left[-\frac{1}{2}e^{-r^2}\right]_0^{\infty} = \pi. \qquad \therefore I=\sqrt{\pi}.$$

- $\frac{d}{da}I(a) = -\frac{1}{2a}\sqrt{\frac{\pi}{a}}$. ところで,

$$\frac{d}{da}I(a) = \frac{d}{da}\int_{-\infty}^{\infty} e^{-a(x-b)^2}dx = \int_{-\infty}^{\infty}\frac{d}{da}e^{-a(x-b)^2}dx$$

$$= -\int_{-\infty}^{\infty}(x-b)^2 e^{-a(x-b)^2}dx.$$

$$\therefore \int_{-\infty}^{\infty}(x-b)^2 e^{-a(x-b)^2}dx = \frac{1}{2a}\sqrt{\frac{\pi}{a}}.$$