Ch.1 確率.

§1.1 事象と確率.

- 試行：不確かさを伴う実験，観測等を行うこと.
- Ω : 標本空間
- 事象：起こりうる結果の集まり.

Def. (σ-field)

Ω : set. $\mathcal{F} \subseteq 2^{\Omega}$: σ-field

$\overset{\text{def.}}{\Longleftrightarrow}$ (1) $\emptyset \in \mathcal{F}$

(2) $A \in \mathcal{F} \Rightarrow A^c \in \mathcal{F}$

(3) $A_n \in \mathcal{F} \Rightarrow \bigcup_{n=1}^{\infty} A_n \in \mathcal{F}$.

$A \in \mathcal{F}$: 可測集合 (m'ble set)

$(\Omega, \mathcal{F})$: 可測空間 (m'ble sp.) $\square$

Def. (probability)

$(\Omega, \mathcal{F})$: m'ble sp. $P : \mathcal{F} \to \mathbb{R}$: probability meas.

$\overset{\text{def.}}{\Longleftrightarrow}$ (1) $P(A) \geq 0$ for all $A \in \mathcal{F}$.

(2) $P(\Omega) = 1$.

(3) $\{A_i\}$: disjoint $\Rightarrow P(\bigsqcup_{i=1}^{\infty} A_i) = \sum_{i=1}^{\infty} P(A_i)$. $\square$

Prop. (1) $P(A^c) = 1 - P(A)$

(2) $A \subseteq B \Rightarrow P(A) \leq P(B)$

(3) $P(A \cap B) \leq \min\{P(A), P(B)\}$

(4) $P(A \cup B) = P(A) + P(B) - P(A \cap B)$. $\square$

§1.2 条件付き確率と事象の独立性

Def. (条件付き確率)

$A, B \in \mathcal{F}$, $P(B) > 0$.

$$P(A|B) := \frac{P(A \cap B)}{P(B)}$$

を, Bを与えたときのAの**条件付き確率**という. $\square$

Prop. $P(A_1 \cap \cdots \cap A_n) = P(A_n | A_1 \cap \cdots \cap A_{n-1}) \cdots\cdots P(A_2 | A_1) P(A_1)$ $\square$

Prop. (全確率の公式)

$\{B_n\}_{n=1}^{\infty} \subseteq \mathcal{F}$: disjoint. $P(B_n) > 0$. $\bigsqcup_{n=1}^{\infty} B_n = \Omega$.

$\Rightarrow A \in \mathcal{F}$, $P(A) = \sum_{n=1}^{\infty} P(A|B_n) P(B_n)$. $\square$

pf. $A = A \cap \Omega = A \cap \left(\bigsqcup_{n=1}^{\infty} B_n\right)$ から. $\blacksquare$

Prop. (Bayesの定理)

$\{B_n\}_{n=1}^{\infty} \subseteq \mathcal{F}$: disjoint. $P(B_n) > 0$. $\bigsqcup_{n=1}^{\infty} B_n = \Omega$.

$\Rightarrow A \in \mathcal{F}$,

$$P(B_k | A) = \frac{P(A|B_k)P(B_k)}{\sum_{n=1}^{\infty} P(A|B_n)P(B_n)}.$$

$(k = 1, 2, \cdots)$ $\square$

Def. (独立)

$A, B \in \mathcal{F}$: 独立 (indep.) $\overset{\text{def.}}{\Longleftrightarrow} P(A \cap B) = P(A)P(B)$. $\square$

§1.3 発展的事項.

Def. (単調列)

$\{A_n\}_{n=1}^{\infty} \subseteq \mathcal{F}$: 単調増大列 $\overset{\text{def.}}{\Longleftrightarrow} A_n \subseteq A_{n+1}$ $(n = 1, 2, \cdots)$

$\{A_n\}_{n=1}^{\infty} \subseteq \mathcal{F}$: 単調減少列 $\overset{\text{def.}}{\Longleftrightarrow} A_n \supseteq A_{n+1}$ $(n = 1, 2, \cdots)$

$\square$

Prop. (1) $\{A_n\}_{n=1}^{\infty} \subseteq \mathcal{F}$: 単調増大列

$$\Rightarrow P\left(\bigcup_{n=1}^{\infty} A_n\right) = \lim_{n \to \infty} P(A_n) \quad \text{(確率の連続性)}$$

(2) $\{A_n\}_{n=1}^{\infty} \subseteq \mathcal{F}$, $P\left(\bigcup_{n=1}^{\infty} A_n\right) \leq \sum_{n=1}^{\infty} P(A_n)$

(確率の劣加法性) $\square$

Prop. (Bonferroniの不等式)

$$A_n \in \mathcal{F} \quad (n=1,2,\cdots),\quad P\left(\bigcap_{n=1}^{\infty} A_n\right) \geqq 1 - \sum_{n=1}^{\infty} P(A_n^c). \qquad \square$$

pf. $P\left(\bigcup_{n=1}^{\infty} A_n^c\right) \leq \sum_{n=1}^{\infty} P(A_n^c)$.

$$P\left(\bigcup_{n=1}^{\infty} A_n^c\right) = P\left(\left(\bigcap_{n=1}^{\infty} A_n\right)^c\right) = 1 - P\left(\bigcap_{n=1}^{\infty} A_n\right). \qquad \blacksquare$$

Prop. (Borel-Cantelliの補題)

(1) $\{A_n\}_{n=1}^{\infty} \subseteq \mathcal{F}$, $\sum_{n=1}^{\infty} P(A_n) < \infty \Rightarrow P\left(\limsup_{n\to\infty} A_n\right) = 0$

$\left(\limsup_{n\to\infty} A_n := \bigcap_{n=1}^{\infty} \bigcup_{k=n}^{\infty} A_k\right)$

(2) $\{A_n\}_{n=1}^{\infty} \subseteq \mathcal{F}$: indep., i.e., $P\left(\bigcap_{i=1}^{n} A_i\right) = \prod_{i=1}^{n} P(A_i)$, $\forall n$.

$\sum_{n=1}^{\infty} P(A_n) = \infty \Rightarrow P\left(\limsup_{n\to\infty} A_n\right) = 1$.

pf. (1) $B_n := \bigcup_{k=n}^{\infty} A_k$ とおくと, $\{B_n\}_{n=1}^{\infty} \subseteq \mathcal{F}$ は単調減少列.

$$P\left(\limsup_{n\to\infty} A_n\right) = P\left(\bigcap_{n=1}^{\infty} B_n\right) = P\left(\lim_{n\to\infty} B_n\right) = \lim_{n\to\infty} P(B_n)$$

$$= \lim_{n\to\infty} P\left(\bigcup_{k=n}^{\infty} A_k\right) \leq \lim_{n\to\infty} \sum_{k=n}^{\infty} P(A_k) = 0 \quad \left(\because \sum_{n=1}^{\infty} P(A_n) < \infty\right)$$

(2) $\left(\limsup_{n\to\infty} A_n\right)^c = \liminf_{n\to\infty} A_n^c$. $P\left(\liminf_{n\to\infty} A_n^c\right) = 0$ を示す.

$$P\left(\liminf_{n\to\infty} A_n^c\right) = P\left(\bigcup_{n=1}^{\infty} \bigcap_{k=n}^{\infty} A_k^c\right)$$

$C_n := \bigcap_{k=n}^{\infty} A_k^c$ とすると. $\{C_n\}_{n=1}^{\infty} \subseteq \mathcal{F}$ は単調減少列.

$$P\left(\bigcup_{n=1}^{\infty} \bigcap_{k=n}^{\infty} A_k^c\right) = P\left(\bigcup_{n=1}^{\infty} C_n\right) = P\left(\lim_{n\to\infty} C_n\right) = \lim_{n\to\infty} P(C_n)$$

$$= \lim_{n\to\infty} P\left(\bigcap_{k=n}^{\infty} A_k^c\right) \leq P\left(\bigcap_{k=n}^{N} A_k^c\right) \quad (N:十分大)$$

$$= \prod_{k=n}^{N} P(A_k^c) = \prod_{k=n}^{N} (1 - P(A_k))$$

$$\leq \prod_{k=n}^{N} \exp(-P(A_k)) = \exp\left(-\sum_{k=n}^{N} P(A_k)\right) \xrightarrow[N\to\infty]{} 0. \quad \blacksquare$$